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DISCRETE ANALYSIS METHOD FOR SUSPENSION BRIDGES

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Abstract. In the calculation of suspension bridges, the geometrically non-linear behaviour of the parabolic cable is the main problem. The linear methods of analysis suit only for small spans. A geometrically non-linear continual model is especially useful for classical loading cases – a uniformly distributed load on the whole or a half span. But the modern traffic models consist of concentrated and uniformly distributed loads. The discrete model of a suspension bridge allows us to apply all kinds of loads, such as distributed or concentrated ones. The simplest suspension bridge consists of a geometrically non-linear cable, connected by hangers with an elastic linear stiffening girder. Depending on the load case, the hangers may be unequally loaded; thus the cable may also be loaded by unequal concentrated forces. The assumptions of the discrete method described here are: linear elastic strain-stress dependence on the material and absence of horizontal displacements of hangers. Hangers elongation is taken into account. Some comparative numerical examples are presented.

Keywords: cables, continual analysis, discrete analysis, geometric non-linearity, suspension bridge.

1. Introduction

Classical treatment of suspension bridges is presented in [1, 2]. A generalised method of continuous modelling of different prestressed cable structures was proposed in [3, 4]. It includes plane structures and spatial networks and proceeds from geometrically non-linear equilibrium conditions and equations of deformation compatibility [3–5]. A peculiarity of this method is an immediate inserting of displacements of cable supports into the generalised equations of deformation compatibility.

Recent publications in the field of suspension bridges consider mainly the wind-induced dynamic processes and specific problems of the bridge elements design [6]. A thorough review of literature in this field is given in the handbook [7]. However, only linear models have been considered [7]. More recent papers on suspension bridges consider the application of the standard finite element method [8].

In this paper the analysis of the suspension bridges is carried out using non-linear equilibrium conditions and generalised equations of deformation compatibility, taking into account the actual boundary conditions.

2. Discrete model for elastic cable

If the cable is loaded by a uniformly distributed load, then it takes the parabolic form. In reality the cable is loaded by concentrated forces, and it takes the form of the string polygon. Then it may be regarded as a geometrically non-linear rod without bending stiffness. If the applied forces are equal and uniformly situated, then the nodes of this polygon are on the parabolic line.

The initial state of equilibrium of the cable loaded by a concentrated load is shown in Fig 1.

From the equilibrium considerations of forces we may write for every node [4]:

$$H_0\left(\frac{z_{i-1}-z_i}{a_{i-1}}+\frac{z_{i+1}-z_i}{a_i}\right)+F_{0i}=0, \qquad (1)$$

where H_0 is the initial force horizontal component of the

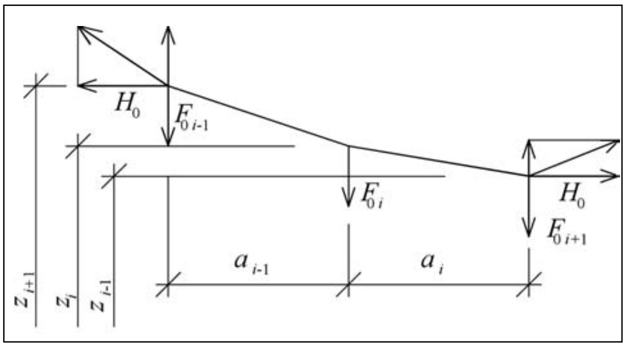

Fig 1. A discrete scheme of the cable in the state of equilibrium

cable (cable force), z_{i-1}, z_i, z_{i+1} are the initial vertical coordinates of the cable, a_{i-1}, a_i are the horizontal distance between cable nodes, and F_{0i} is the initial external force.

From Eq (1) we have

$$z_i = \frac{1}{1+\dfrac{a_i}{a_{i-1}}}\left(z_{i-1}\frac{a_i}{a_{i-1}} + z_{i+1} + \frac{F_{0i}a_i}{H_0} \right), \qquad (2)$$

and

$$z_{i+1} = z_i - a_i\left(\frac{z_{i-1}-z_z}{a_{i-1}} + \frac{F_{0i}}{H_0} \right). \qquad (3)$$

Eqs (2) and (3) give us the coordinates of the string polygon, if the horizontal force H_0 is known. When the supporting points of the cable are on the same level ($z_0 = z_{n+1}$), then

$$H_0 = V_0\frac{a_0}{z_1 - z_0} = \frac{a_0\sum_{i=1}^{n} F_{0i}(l - x_i)}{l(z_1 - z_0)}, \qquad (4)$$

where $V_0 = \dfrac{\sum_{i=1}^{n} F_{0i}(l - x_i)}{l}$ is the vertical support reaction from the initial loads F_{0i} and l is the cable length of the horizontal projection.

For a cable which has supporting nodes on different level we may calculate H_0 as

$$H_0 = \frac{a_0\sum_{i=1}^{n} F_{0i}(l - x_i)}{l(z_1 - z_0) - a_0(z_0 - z_{n+1})}. \qquad (5)$$

Here the initial form of the string polygon is described by three coordinates z_0, z_1, z_{n+1}. Instead of z_1 you may use another known value of the coordinate z_i.

By the action of the temporary loads ΔF_i (Fig 2), the equilibrium equation for the node i is expressed as

$$H\left(\frac{z_{i-1}-z_i}{a_{i-1}} + \frac{z_{i+1}-z_i}{a_i} + \frac{w_{i-1}-w_i}{a_{i-1}} + \frac{w_{i+1}-w_i}{a_i} \right) + F_i = 0, \qquad (6)$$

where w_{i-1}, w_i, w_{i+1} are vertical displacements, H – the thrust from temporary and initial load and $F_i = F_{0i} + F_i$ is the whole concentrated load in node i.

From Eq (6) we get an expression for the vertical displacement

$$w_i = \frac{1}{1+\dfrac{a_i}{a_{i-1}}}\left[w_{i-1}\frac{a_i}{a_{i-1}} + w_{i+1} + \frac{(z_{i-1}-z_i)a_i}{a_{i-1}} + (z_{i+1}-z_i) + \frac{F_i a_i}{H} \right]. \qquad (7)$$

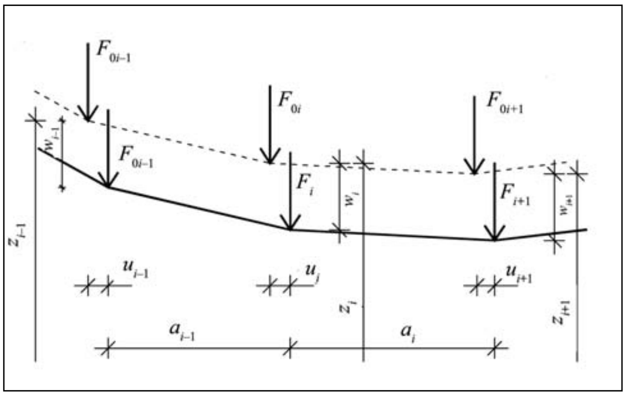

Fig 2. Deformation of the cable under additional load

There are two unknown parameters in Eq (7): w_i and H. Thus we need another equation for calculating them. For this purpose, it is possible to use the principle of minimum total potential energy [3]

$$\sum_{i=1}^{n} F_i w_i - U = 0, \qquad (8)$$

where U is the strain energy of deformed structure. Here, we use the compatibility condition of the relative elongation of the cable [1]. The relative elongation of the cable is expressed as

$$\varepsilon_i = \frac{1}{1+\left(\dfrac{z_{i+1}-z_i}{a_i}\right)^2}\left[\frac{u_{i+1}-u_i}{a_i} + \frac{w_{i+1}-w_i}{a_i}\left(\frac{z_{i+1}-z_i}{a_i} + \frac{w_{i+1}-w_i}{2a_i} \right) \right] \qquad (9)$$

and from the condition of linear deformation

$$\varepsilon_i = \frac{H-H_0}{EA}\sqrt{1+\left(\frac{z_{i+1}-z_i}{a_i}\right)^2}, \qquad (10)$$

where EA is the stiffness of the cable in tension and u_i, u_{i+1} are the horizontal displacements of cable nodes.

Taking into account (9) and (10), this compatibility condition may be presented as

$$\frac{u_{i+1}-u_u}{a_i} = \frac{H-H_0}{EA}\left[1+\left(\frac{z_{i+1}-z_i}{a_i}\right)^2 \right]^{3/2} - \frac{w_{i+1}-w_i}{a_i}\left(\frac{z_{i+1}-z_i}{a_i} + \frac{w_{i+1}-w_i}{2a_i} \right). \qquad (11)$$

Horizontal displacements of the internal nodes u_i, u_{i+1} may be eliminated by means of summation of the equations of deformation compatibility (11) and after replacing

$$\sum_{i=0}^{n}\left(u_{i+1}-u_i\right)=u_{n+1}-u_0 \qquad (12)$$

we may write the Eq (11) in the form

$$\frac{H-H_0}{EA}\left\{\sum_{i=0}^{n} a_i\left[1+\left(\frac{z_{i+1}-z_i}{a_i}\right)^2\right]^{\frac{3}{2}}-\frac{EA}{H-H_0}\left(u_{n+1}-u_0\right)\right\}=$$
$$\sum_{i=1}^{n}\left(w_{i+1}-w_i\right)\left(\frac{z_{i+1}-z_i}{a_i}+\frac{w_{i+1}-w_i}{2a_i}\right). \qquad (13)$$

Solution of the system of non-linear equations (6) and (13) enables us to calculate all the displacements w_i and H by the given initial cable form and boundary conditions u_0, u_{n+1}.

3. Discrete model for elastic cable with the stiffening girder

The scheme of a girder-stiffened suspension bridge is presented in Fig 3.

The initial vertical load F_{0i} is fully balanced by the cable and prestresses it. For calculating the initial cable force H_0 we may use expressions (4) or (5). Part of the additional load P is balanced by the cable and the rest of it is balanced by the stiffening girder. The equation that describes the deflection of the girder can be written as:

$$E_bI_bw(x)=E_bI_bw_0+E_bI_b\varphi_0x-\mathsf{M}\frac{(x-a)^2}{2}\cdot H(x-a)+$$
$$\mathsf{F}\frac{(x-b)^3}{6}\cdot H(x-b)+\mathsf{p}\frac{(x-c)^4}{24}\cdot H(x-c)-$$
$$\mathsf{p}\frac{(x-d)^4}{24}\cdot H(x-d), \qquad (14)$$

where $E_b\,I_b$ is the rigidity of the stiffening girder in bending, w_0 – the vertical displacement at the first point of girder, φ_0 – the angle of rotation at the first point of the girder, *a, b, c, d* – the coordinates of the points of the force application and $H(x)$ is the Heaviside's function.

Eq (14) can be used for calculating deflection from the sum of applied external concentrated moments M, forces F, and from the uniformly distributed load p.

In case of vertical pylons, the horizontal displacements of the supporting nodes of the cable may be presented as

$$u_0=u_{n+1}=\frac{\left(H-H_0\right)l}{EA\cos^3\alpha}, \qquad (15)$$

where l is the length of the anchor cable and α is the angle of inclination of this cable.

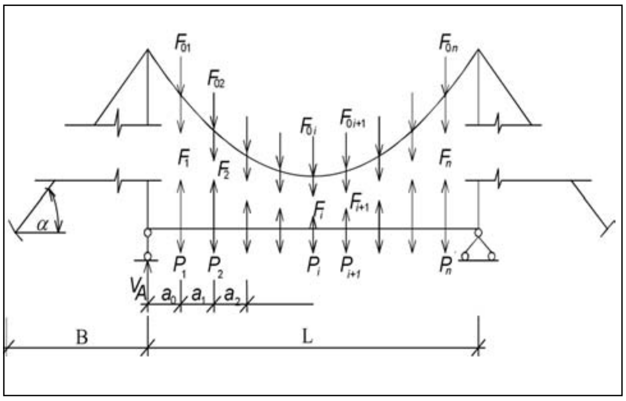

Fig 3. A suspension bridge model

Let us consider only the stiffening girder. Using Eq (14) we can write for every hanger joint and support point B:

$$w_m=w_0+\varphi_0x_m+\sum_{i=1}^{m-1}F_i\frac{\left(x_m-x_i\right)^3}{6E_bI_b}+V_A\frac{x_m^3}{6E_bI_b}-$$
$$\mathsf{M}\frac{(x-a)^2}{2}\cdot H(x-a)+\mathsf{F}\frac{(x-b)^3}{6}\cdot H(x-b)+ \qquad (16)$$
$$\mathsf{p}\frac{(x-c)^4}{24}\cdot H(x-c)-\mathsf{p}\frac{(x-d)^4}{24}\cdot H(x-d),$$

where F_i is the internal force in the hangers and V_A – the vertical support reaction.

We obtain $n+1$ linear equations for calculating F_i, but there are $n+2$ unknown parameters: F_1, F_2 ... F_n, V_A and φ_0. An extra equation can be written from the moment equilibrium condition upon support B, as follows:

$$\sum_{i=1}^{n}F_i\left(L-x_i\right)+V_AL+M_P=0\,, \qquad (17)$$

where M_P is the moment of the external forces upon support B, and L – the span of the suspension bridge.

It became evident that it is reasonable to converge all linearly interdependent components into a uniform linear equation system, which thereafter will be dependent on the cable's internal force H. Thus, the solution is reduced to the search of such H, when placing the displacements calculated from the linear equation system into the expression linking the elongation of the cable and the displacements (13) and the H found in its solution equals the H used for compiling the linear equation system. As an algorithm, such a system could be described as in Fig 4.

The matrix of the linear equation system of the corresponding system can be presented as follows:

$$\begin{vmatrix}
A_{1,1} & A_{1,2} & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
A_{2,1} & A_{2,2} & A_{2,3} & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & A_{3,2} & A_{3,3} & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & \dots & A_{n-1,n-1} & A_{n-1,n} & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \dots & A_{n,n-1} & A_{n,n} & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & \dots & 0 & 0 & B_{1,1} & B_{1,2} & B_{1,3} & \dots & B_{1,n-1} & B_{1,n} & B_{1,n+1} & B_{1,n+2} \\
0 & -1 & 0 & \dots & 0 & 0 & B_{2,1} & B_{2,2} & B_{2,3} & \dots & B_{2,n-1} & B_{2,n} & B_{2,n+1} & B_{2,n+2} \\
0 & 0 & -1 & \dots & 0 & 0 & B_{3,1} & B_{3,2} & B_{3,3} & \dots & B_{3,n-1} & B_{3,n} & B_{3,n+1} & B_{3,n+2} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & \dots & -1 & 0 & B_{n-1,1} & B_{n-1,2} & B_{n-1,3} & \dots & B_{n-1,n-1} & B_{n-1,n} & B_{n-1,n+1} & B_{n-1,n+2} \\
0 & 0 & 0 & \dots & 0 & -1 & B_{n,1} & B_{n,2} & B_{n,3} & \dots & B_{n,n-1} & B_{n,n} & B_{n,n+1} & B_{n,n+2} \\
0 & 0 & 0 & \dots & 0 & 0 & B_{n+1,1} & B_{n+1,2} & B_{n+1,3} & \dots & B_{n+1,n-1} & B_{n+1,n} & B_{n+1,n+1} & B_{n+1,n+2} \\
0 & 0 & 0 & \dots & 0 & 0 & D_1 & D_2 & D_3 & \dots & D_{n-1} & D_n & 0 & D_{n+2}
\end{vmatrix}
\bullet
\begin{vmatrix} w_1 \\ w_2 \\ w_3 \\ \dots \\ w_{n-1} \\ w_n \\ F_1 \\ F_2 \\ F_3 \\ \dots \\ F_{n-1} \\ F_n \\ \varphi_0 \\ V_A \end{vmatrix}
=
\begin{vmatrix} C_1 \\ C_2 \\ C_3 \\ \dots \\ C_{n-1} \\ C_n \\ C_{F,1} \\ C_{F,2} \\ C_{F,3} \\ \dots \\ C_{F,n-1} \\ C_{F,n} \\ C_{F,n+1} \\ C_D \end{vmatrix}$$

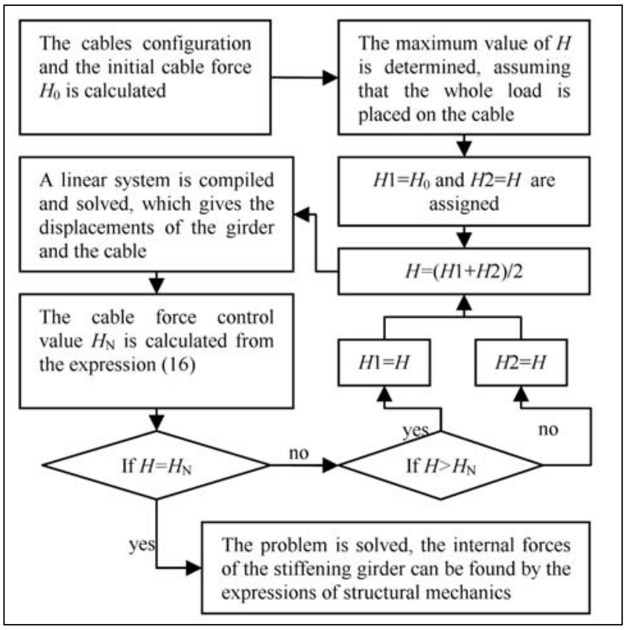

Fig 4. Algorithm of equation system

Here the matrix components $A_{i,j}$ and free term C_i, derived from (6) can be expressed as:

$$A_{i,i-1} = a_i H; \quad A_{i,i} = H\left(a_{i-1} + a_i\right); \quad A_{i,i+1} = a_{i-1} H;$$

$$C_i = -H\left(z_i w_{i-1} - (z_{i-1} + z_i) w_i + z_{i-1} w_i\right) - F_i \cdot a_{i-1} a_i. \quad (18)$$

The coefficients $B_{i,j}$ and $C_{F,i}$ have been derived from the universal equation of the girder's elastic curve (16) and can be presented as:

if $i > j$; $1 \le i \le n$ and $1 \le j \le n$, then

$$B_{i,j} = \frac{\left(x_i - x_j\right)^3}{6 E_B I_B},$$

otherwise $B_{i,j} = 0$,

and for $j > n$

$$B_{i,n+1} = x_i; \quad B_{i,n+2} = \frac{x_i^3}{6EI}. \quad (19)$$

The free term $C_{F,i}$ depends on the specific load situation of the bridge, and contains all these coefficients of the universal equation of the elastic curve, which do not contain the sought deformations and internal forces:

$$C_{F,i} = \sum_{k=1}^{s} F_k \frac{(x - b_k)^3}{6} \cdot H(x - b_k) +$$
$$\sum_{l=1}^{t} p_l \frac{(x - c_l)^4}{24} \cdot H(x - c_l) - \sum_{l=1}^{t} p_l \frac{(x - d_l)^4}{24} \cdot H(x - d_l).$$

In case of a bridge loaded with uniform loading p, the free term is in the following form:

$$C_{F,i} = \frac{p x_i^4}{24 E_B I_B}. \quad (20)$$

The coefficients of the last row of the matrix are derived from the equilibrium condition of the moments as related to the bridge support B, and are presented as

$$D_i = \left(l - x_i\right); \quad D_{n+2} = l; \quad (21)$$

and the free term depends on the loads placed on the bridge; in case of a uniformly loaded bridge, the free term is

$$C_D = \frac{p l^2}{2}. \quad (22)$$

4. Numerical examples

Let us have suspension bridges with the following parameters, used for provisional design of the bridge for Saaremaa Fixed Link (Table 1).

A comparison of maximum deflections from the whole

Table 1. Parameters of suspension bridges

Span, m	600	780	960	1080	1200
Sag of the cable, m	75	97,5	120	135	150
Side span, m	250	325	400	450	500
Moment of inertia of the stiffening girder $I_b \times 104$ cm^4	1269	2488	2964	4524	5896
Young's modulus of the stiffening girder E_b, GPa	210	210	210	210	210
Cross-section area of the cable A, cm^2	768	1092	1458	1738	2008
Young's modulus of the cable E, GPa	170	170	170	170	170
Initial load p_0, kN/m	30,4	38,2	43,3	50,0	54,8
Additional weight of the bridge deck p_1, kN/m	14,4	14,1	14,4	14,4	14,4
Whole or half-span traffic load p_2, kN/m	31,1	30,7	30,4	30,0	29,6

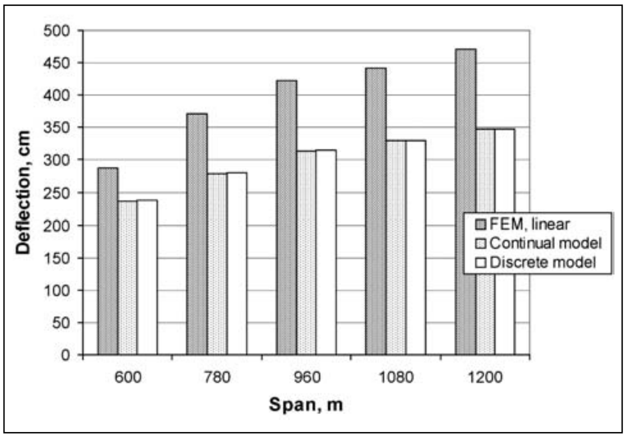

Fig 5. Deflections from the whole span uniformly distributed traffic load

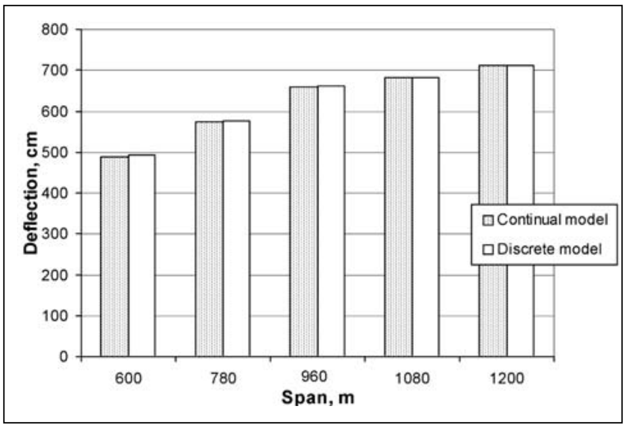

Fig 6. Deflections from the half span uniformly distributed traffic load

and half span traffic load p_2 are shown in Figs 5 and 6. Thus the linear principle of superposition of the load is not valid for suspension bridges; the deflection caused by traffic load is calculated by the expression:

$$w(p_2) = w(p_0 + p_1 + p_2) - w(p_0 + p_1). \qquad (23)$$

The continual method used here is described in [3, 5]. The linear method is the usual linear finite elements method (FEM). In case of a uniformly distributed half-span load, the linear FEM method gave unrealistic displacements, and these results are not shown in Fig 6.

5. Conclusions

1. In this article, the equations of the discrete method for statical analysis of suspension bridges have been presented. For a numerical solution, a system of non-linear equations has been derived. This method may be used universally for all kinds of loads – uniformly distributed whole and half-span load, concentrated wheel or axial load.

2. Numerical examples show, that in case of the uniformly distributed whole and half-span load, the continual method [3] and the discrete method give practically the same maximum vertical deflection. Vertical displacements in case of uniformly distributed whole span load are approximately 1,3–1,4 times less than those obtained by the linear FEM method. This method is not applicable in case of a uniformly distributed half-span load.

3. If discrete loadings are predicted for the suspension bridge, the discrete calculation method should be used for a preliminary calculation of the bridge instead of continual calculation method or linear FEM.

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